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Finite element modelling of radiation in a nonparticipating medium coupled with conduction and convection heat transfer with moving boundaries

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Abstract *The radiation conduction coupling leads to particular problems due to computation time and high heat fluxes. Because of the hemispheric nature of the radiation, it is difficult to take into account symmetric or periodic conditions for the reduction of the modelled domain. We developed a finite element model of radiative heat transfers between grey diffuse surfaces with a nonparticipating medium with periodic or symmetric boundary conditions. The approaches used to decrease the computation time allowed the modelling of moving radiative surfaces. We introduced this model into a finite element convection diffusion code in order to simulate heat transfers in an electrical rotating engine. The main originality of this study lies in the use of periodic radiative conditions with moving surfaces and in the use of a method which is not based on the isothermal approximation.*

Nomenclature

q_e = angle between $r_{e\text{-}n}$ and n_e
q_n = angle between $r_{e\text{-}n}$ and n_n (normal to G_n)
n_e = normal vector to G_e at dG_e
N_{elt} = total number of elements
T = temperature (°C)
dG_e = elementary surface G_e
dT = first variational of T
G_e = surface boundary element
W_e = domain element
s = radiation constant s=5, 67 10^{-8} ($W/m^2.K^4$)
N_{ev} = number of element viewed by G_e
R_e = radiosity of dG_e (W/m^2)
$r_{e\text{-}n}$ = segment joining dG_e and dG_n
q_e = angle between n_e and $r_{e\text{-}n}$ (radians)
j_x = Jacobian determinant multiplied by Gaussian weight
$<N_i>$ = interpolation function
x_l, x_k = Gaussian point coordinates in G_e and G_n in the reference.
S_e = surface of element G_e
e_e, a_e, r = emissivity, absorptivity and reflexivity of dG_e
$[k_{e\ h}]$ = elementary matrix of boundary convection conditions
$[k_{ec}]$ = elementary matrix of diffussion
$[k_{er}]$ = elementary matrix of boundary emitted radiation
$\{f_{e\ h}\}$ = elementary vector of boundary convection condition
$\{f_{ei}\}$ = elementary vector of imposed flux boundary condition
$\{f_{er}\}$ = elementary vector of received boundary radiation condition

Physical model

The physical phenomenon taken into account is the radiative transfer which occurs at the boundary of an opaque medium (see Figure 1). The radiative

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surfaces are assumed grey and diffuse and separated by a transparent medium. This radiative transfer is coupled to a diffusive transfer. The diffusive transfer is governed by the heat equation.

$$\rho C_p \left(\frac{\partial T}{\partial t} + \vec{v}.\vec{\nabla} T \right) = \lambda \Delta T + \dot{Q} \tag{1}$$

The radiative transfers occur as flux boundary conditions. The radiative flux is expressed according to the temperature of the surface and the radiosity of the viewed surfaces (2).

$$\phi_{er} = \varepsilon_e \sigma T_e^4 - \varepsilon_e \sum_{n=1}^{Nev} \left(\int_{\Gamma_n} R_n \frac{Cos\theta_e Cos\theta_n}{\pi r_{e-n}^2} d\Gamma_n \right) \tag{2}$$

The radiosities are linked to temperatures via the radiative transfer equation expressed in (3) for an elementary surface Γ_e in a diffuse grey enclosure.

$$R_e = \varepsilon_e \sigma T_e^4 + \rho_e \sum_{n}^{Nev} \int_{\Gamma_n} R_n dF_{n-e} d\Gamma_n \tag{3}$$

Both equations (1) and (3) are strongly coupled.

Finite element formulation of radiation conduction coupling

The finite element formulation used is a Galerkin formulation. It is widely described in the literature for example by Touzot and Dhatt (1981) and Comini *et al.*, (1994). If the medium is transparent, radiation occurs as a flux boundary condition. The radiative flux creates solicitations at the nodes of the surfaces which surround the opaque domain. The computation of the solicitations characterizes the different approaches used in the literature. They all compute the radiative fluxes. Computations for grey and diffuse surfaces have been described in the literature. They solve the radiative transfer equation. The unknowns are the nodal radiosity values or the elementary radiative fluxes with respect to temperature. A most successful method is the isothermal zone one. In this approach, the surfaces are divided into isothermal zones. The temperature, the radiative properties, the emitted, incident and reflected radiative fluxes are constant on each zone. This approach is well suited to numerical computation because the boundary elements of the mesh are easily identified to these isothermal zones.

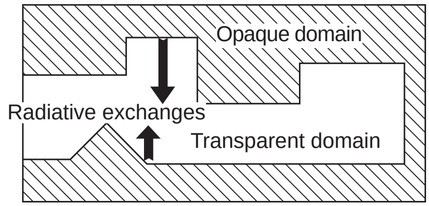

Figure 1.
Physical problem

This approach often requires very fine meshes in the high temperature or strong gradient areas. The approximation on which they rely is questioned. The illumination of a zone by another, namely the radiative flux emitted by an element and received by another one, may vary strongly. The variation exists even when the sources are isothermal because the radiative fluxes received by an elementary surface of the element is inversely proportioned to the square of the distance, which separates the elementary surface from the source. This isothermal approximation is penalising in several cases. The faults of these methods have been pointed out by Reddy and Murty (1978) from 1978 on and more recently by Daurelle (1992, 1994); Daurelle *et al.* (1994) and Lobo and Emery (1995).

The approach we use does not rely on the isothermal approximation. The radiative fluxes are averaged on elements but expressed according to the radiosities directly into the finite element formulation of the heat equation. The radiosities are computed from temperatures by applying a finite element formulation to the radiative transfer equation. The iterative process consists of the solutions of radiative and then heat transfer equations up to convergence. This formulation noted D.I. for direct integration is more precise because it takes into account variations of the emitted, incident and reflected fluxes. However, the D.I. computation times are noticeably more important than those of an isothermal zone method.

In this work, the precise integration method has been associated with a formulation closed to the isothermal zone method.

Our approach adapts the computational effort to the treated configuration. The fluxes are expressed either via the zone method or via the D.I. method according to the treated element couple. We use a criterion based on the part of the elementary radiative flux compared with the total received flux in order to determine the choice for the method. This criterion does not involve any additional computation cost.

Elementary parameters

The energy equation discretized via finite elements is written after part integration under the following weak integral form:

$$I = \sum_{e=1}^{\text{N élémnts}} I_e = 0$$

$$I_e = \int_{\Omega e} \delta T \left(\rho C_p \frac{\partial T}{\partial t} \right) d\Omega_e + \int_{\Omega_e} \tag{4}$$

$$+ \vec{\nabla} \delta T (K_T \vec{\nabla} T) d\Omega_e - \int_{\Gamma_e} \delta T \phi_{\text{bounary}} d\Gamma - \int_{\Omega_e} \delta T \vec{V} . \vec{\nabla} T d\Omega_e$$

The surface radiation terms occur as imposed flux boundary conditions on the limits of the opaque domain. The term I_{er} has to be added to the I_e integral form to take into account the radiation effect.

$$I_{er} = -\int_{\Gamma e} \delta T \phi_{er} d\Gamma e \qquad (5)$$

The net radiative flux Φ_{er} is expressed on an element under the form (2). If the surface Γ_e is grey and diffuse, then the flux can be expressed without the summation on the viewed elements.

$$\phi_{er} = \frac{\varepsilon_e}{1 - \varepsilon_e}\left(\sigma T_e^4 - R_e\right) \qquad (6)$$

To decrease the non linearity of the finite element system, the elementary integral form I_{er} has been divided into two integral forms. One corresponds to the emitted radiative flux I_{ere} and the other to the received radiative flux I_{err}.

$$I_{er} = I_{ere} + I_{err} \qquad (7)$$

The term of the emitted radiation is expressed according to temperature under the form

$$I_{ere} = -\int_{\Gamma_e} E \delta T \sigma T_e^4 d\Gamma \qquad (8)$$

Where the expression of E is

$$E = \frac{\varepsilon}{1 - \varepsilon}$$

for grey surfaces and for black surfaces

$$E = 1$$

The elementary vector of the nodal residues, which corresponds to the emitted radiation, is expressed according to the nodal temperature $\{Te\}$ and to an elementary matrix of the emitted radiation $[k_{re}]$

$$I_{ere} = \langle \delta T_e \rangle [k_{re}] \{T_e\} \qquad (9)$$

With the Gaussian integration, the expression of $[k_{re}]$ becomes

$$\left[k_{re}\left({T_e}^3\right)\right] = -\sum_{k=1}^{P_g} E\sigma \left(\sum_{i=1}^{m} N_i(\xi_k) T_{ei}\right)^3 j_\xi(\xi_k) \langle N(\xi_k) \rangle \{N(\xi_k)\} \qquad (10)$$

The term of the received radiation is expressed according to the radiosity under the following integral form

$$I_{err} = \sum_{n}^{Nev} \int_{\Gamma e} \delta T \int_{\Gamma n} R_n \frac{\cos(\theta_e)\cos(\theta_n)}{r_{e-n}^2} d\Gamma_n d\Gamma \qquad (11)$$

If the surface of G_e is assumed grey and diffuse, the integrals are written under the following form

$$I_{err} = -\int_{\Gamma e} \delta T_e \frac{\varepsilon_e}{1-\varepsilon_e} R_e d\Gamma_e \tag{12}$$

The introduction of the interpolation function leads to the expressions of the elementary vectors of the received radiation. They depend on the nature of the surface (grey or black). If the surfaces are grey diffuse the vector is expressed simply

$$\{f_{er}\} = \int_{\Gamma_e} \langle \delta T_e \rangle \frac{\varepsilon_e}{1-\varepsilon_e} R_e d\Gamma e \tag{13}$$

However, if the element is black the expression of the vector is a sum of all the viewed elements of elementary vectors $\{f_{er}\}_n$.

$$\{f_{er}\} = \sum_{n}^{Nev} \{f_{er}\}_n = \sum_{n}^{Nev} \int_{\Gamma e} \langle \delta T_e \rangle \int_{\Gamma_n} R_n \frac{\cos(\theta_e)\cos(\theta_n)}{r_{e-n}^2} d\Gamma_n d\Gamma e \tag{14}$$

There are two possible expressions of the vectors. A classical approach of the isothermal zone uses the form factors. The second option integrates via a Gaussian method to take into account the temperature variations and the lighting on the elements. With the isothermal approximation the vector is written

$$\{f_{er}\}_n = \frac{1}{S_e} (F_{e-n} R_{n\ \mathrm{average}}) \sum_{1}^{Pg} \{N_e(\xi_1)\} j_\xi(\xi_1) \tag{15}$$

When integrating via the Gaussian method the vector becomes

$$\{f_{err}(T_n^4)\}_n = \sum_{l=1}^{Pg} j_e(\xi_1)\{N(\xi_1)\} \sum_{k=1}^{Pg} j_n(\xi_k) dF_{e-n}(\xi_1, \xi_1) Ko(\xi_P \xi_1) R_n(\xi_k) \tag{16}$$

The computation times are then proportionate to the number of nodes as is the isothermal approximation. They are also proportionate to the square of the number of Gaussian points. Nevertheless, the description of the flux is more accurate particularly when there is a shadow between the elements. The shadow is taken into account finely with a shadow factor between each Gaussian point $K_0(\xi_e, \xi_n)$.

The isothermal approximation is used for the main coupled elements. The direct integration is applied only between the closest elements. The choice for one or the other computation methods for vector $\{f_{er}\}_n$ is carried out according to the fraction of the total flux received by element Γ_e from Γ_n (Daurelle, 1992).

$$\mathrm{Cr}_{e-n} = \frac{\phi_{\mathrm{ray}_{e-n}}}{\sum\limits_{n}^{Nev} \phi_{\mathrm{ray}_{e-n}}} \tag{17}$$

We impose a value of $\mathrm{Cr}_{e\text{-}n}$ below which the solicitation (13) is expressed with the isothermal approximation. In practice, a 0.05 value provides the best results. It will represent only a very slight part, about 2 per cent, of the treated interactions. The discussion of the pertinent values of $\mathrm{Cr}_{e\text{-}n}$ is presented in Daurelle (1994).

Computation of the nodal values of the radiosities

For the computation of the nodal values of the radiosities, which occur in (3) and (4), we have applied to the radiative balance a Galerkin finite element formulation on each element. The integral form of the radiative transfer equation is written

$$\begin{aligned} I_R = \sum_{e}^{Nelt} \left\{ \int_{\Gamma_e} \delta R_e R_e d\Gamma_e - \int_{\Gamma_e} \delta R_e \sum_{n}^{Nelt} \int_{\Gamma_n} \rho_e R_n dF_{n-e} d\Gamma_n d\Gamma_e \right\} \\ - \left\{ \int_{\Gamma_e} \delta R_e \varepsilon_e \sigma T_e^4 d\Gamma_e \right\} \end{aligned} \tag{18}$$

The elementary integral forms can be divided into three integral forms corresponding to the total radiation (I_{eRt}), to the reflected variation (I_{eRr}) and to the proper emission (I_{eRe}). After introducing the interpolation functions, the integral forms lead to the following expressions

$$I_{eRt} = \langle \delta R_e \rangle \int_{\Gamma_e} \{N_e\}\langle N_e \rangle d\Gamma_e \{R_e\} = \langle \delta R_e \rangle [k_{eRt}]\{R_e\} \tag{19}$$

$$I_{eRe} = -\langle \delta R_e \rangle \int_{\Gamma_e} \{N_e\} \varepsilon_e \sigma T_e^4 d\Gamma_e = -\langle \delta R_e \rangle \{f_{eR}\} \tag{20}$$

$$I_{eRr} = \sum_{n}^{Nelt} I_{eRrn} = \sum_{n}^{Nelt} -\langle \delta R_e \rangle \int_{\Gamma_e} \rho_e \int_{\Gamma_n} \{N_e\}\langle N_n \rangle dF_{n-e} d\Gamma_n d\Gamma_e \{R_n\} \tag{21}$$

$$I_{eRrn} = \langle \delta R_e \rangle [k_{eRr}]_n \{R_n\} \tag{22}$$

The computation of matrix $[k_{eRt}]$ and of vector $\{f_{eR}\}$ has not been detailed because it does not represent any difficulty. The expression of matrix $[k_{eRt}]_n$, which corresponds to the radiation reflected by Γ_e, emitted by Γ_n, is a double integral similar to (15). The matrix is computed for grey diffuse surfaces ($\rho_e \neq 0$). only. There are two options to calculate the matrix. Either one applies the approximation of the zone method to use the form factor or one integrates the expression of the reflected radiation directly. Matrix $[k_{eRr}]_n$ is then expressed by integrating via the Gaussian method.

Global thermal finite element system

When we omit the transport term, the global system is identical to the systems corresponding to the diffusion problems with classical boundary conditions (convection, imposed flux, radiation with an outer medium at a constant temperature). The coupling with radiative interaction conditions is obtained by adding matrices [ker] and vectors $\{f_{er}\}$.

$$\sum_{e}^{N\ ele.}\{[k_{ec}]+[k_{eh}]+[k_{er}]\}\{T\}-\sum_{e}^{N\ ele.}\{\{f_{ei}\}+\{f_{eh}\}+\{f_{er}\}\}=0 \qquad (23)$$

Both methods are used selectively in the computation of vector $\{f_{er}\}_n$ and for the radiosity matrices resulting from D.I. or I.A. The obtained algebraic system is strongly nonlinear. The matrix includes terms in T^3 and the right hand side of the system in T^4. A Newton-Raphson method has been used with an Eulerian implicite scheme.

Application: radiative exchange in the air gap of an electrical engine

Consider the heat transfer in the air gap of an electrical engine (Figure 2). Air, which is considered as a Newtonian fluid, is confined between two concentric annulae. The aspect ratio (thickness of the air gap on the outer radius of the rotor) is less than one. In the developed model, the thermal phenomena (conduction-radiation) are coupled with the convective phenomena. The buoyancy influence is neglected because of the low dimensions compared with the rotation velocities.

The rotor provides the fluid of the air gap with heat. It is due to the temperature rise of the resistive winding. A circuit cooled down by water circulation is placed on the outer part of the stator. As the slots are placed

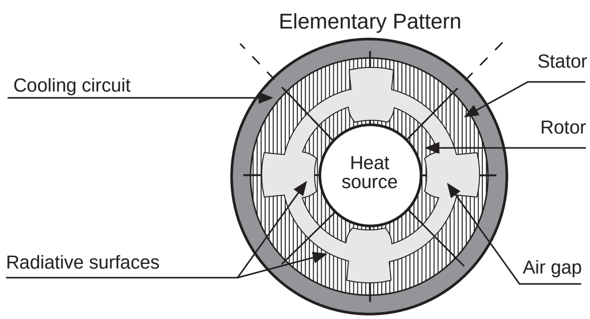

Figure 2. Simulated domain

regularly on the rotor and stator walls, we model only one angular sector. Periodicity conditions have to be introduced on the radial limits of the modelled elementary unit.

Physical problem

It is a 2D study of the periodically unsteady laminar case. The balance equations, which govern the flow, are Navier-Stokes and continuity is solved by a penalty finite element method. The energy equation, which governs heat transfers (1), is coupled with fluid equations via the transport term. As far as buoyancy is neglectable for the rotor velocities, temperature only intervenes in the physical properties in the fluid equation.

Boundary conditions

The boundary conditions of fluid and energy equations are described in Figure 3.

Conditions on circular surfaces. For the fluid problem, a tangential velocity has been imposed to the rotor whereas velocity is nil at the stator. For the thermal system, a constant heat flux has been imposed to the rotor. A constant temperature (50°C) is imposed on the outer surface of the stator to model the cooling flow. The outer walls of the rotor and the inner walls of the stator are grey diffuse. The radiation condition has been applied on both walls and air has been considered as a transparent fluid.

Periodic boundary conditions on the radial surfaces. For each problem as well as for radiation, a periodicity condition has been imposed to the radial sides of the model. The rotor and stator geometries as well as the physical phenomenon (buoyancy is neglected) are periodic. By imposing periodic boundary conditions, it is possible to model only an angular sector of the rotor-stator (set). For this sector, which constitutes the modelled elementary unit (Figure 3), the mass and energy fluxes going out of face 2 have been imposed on face 1 and vice versa.

Considering periodic conditions for the diffusive and mass fluxes does not present any particular difficulty. One only has to transfer the nodal solicitations of the inlet surface onto the outlet surface and vice versa. As far as

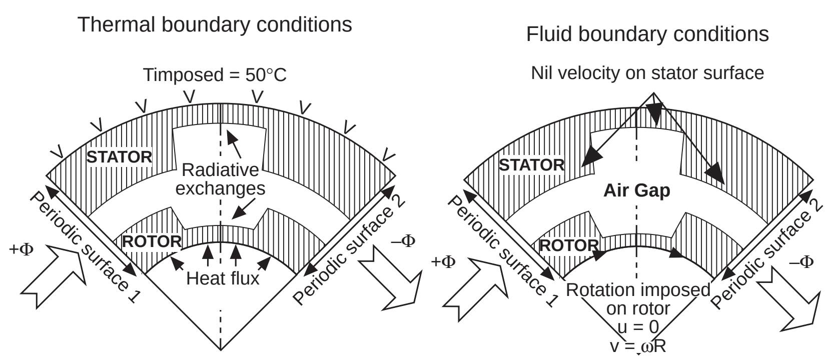

Figure 3.
Boundary conditions

radiative exchanges are concerned, it is more complex to take into account the periodic conditions. Because of the expression of the flux going out of an aperture, the distant action of radiation explains an important part of the radiative surfaces. The radiative flux coming through the aperture is not applied directly but is distributed on the opaque surfaces. Periodic boundary conditions can be taken into account in radiation when the inlet aperture cannot see the outlet aperture. The periodicity condition will then be taken into account during the computation of the form factor. The modelled elementary unit is duplicated virtually to take into account the presence of the previous and following units. For each viewed element Γ_n two fictitious elements $\Gamma_{n'}$ and $\Gamma_{n''}$ have been created. They represent the element Γ_n of the previous and following units. The form factor value between Γ_e and Γ_n is the sum of three factors, one for the viewed Γ_n and the others for $\Gamma_{n'}$ and $\Gamma_{n''}$. The radiative periodic condition leads to a considerable reduction of computation times. Besides, it does not alter the expression of the finite element system because it only changes the form factor computation.

Identification of the fluid medium and the interface

We want to model heat transfer in a domain constituted of two moving media. One medium is entirely transparent to radiation, the other one is opaque. The medium identification is obtained by a scalar parameter F of value 0 at the interface of the negative values in the opaque medium and at the interface of the positive values in the transparent medium. On an element crossed by the interface, the interface can be located by the values at the nodes. For an electrical engine this function is only time dependent.

Medium changes in the energy equation are obtained via thermophysical properties computed from function F at the Gaussian integration point. If F has value –1, the properties are those of the opaque solid. If, on the contrary, F equals 1, the properties are those of air. For the solution of the fluid problem the medium will be taken into account by a change in the viscosity values. If the Gaussian point is in the fluid, its viscosity is the air one. On the contrary, if the element is solid its viscosity is infinite.

Mesh of the moving radiative surfaces

The exchange surface must be remeshed at each evolution and the form factors must be recomputed according to the new interface position (F = 0). As far as the mesh is concerned, there are two possibilities. The first one consists in constructing the whole modelled domain again by remeshing the exchange surfaces and volumes. The advantage of this approach is a good identification of the surface between the media. It is however very expensive in memory and/or computation time. As a matter of fact, a new mesh has to be reconstructed without totally damaging the former one for the interpolation of the nodal values of the unknowns between the new and the former nodes.

In the second approach, only the radiative surfaces are remeshed. The diffusive mesh remains fixed. The change of medium is obtained by a variation of the material properties. The interface is no longer clearly defined in the mesh of the domain but is diffused on an element. On the other hand, it is necessary to identify the radiative surface accurately in order to compute the form factors.

The advantage of this approach is that only the exchange surfaces are remeshed. Interpolation is the only operation, which has been carried out to distribute the solicitations of the radiative flux on the nodes of the diffusive mixed mesh. The radiative fluxes are considered as heat sources. Although it is less accurate than the total remeshing method, this approach enables to save memory and computational time. This is the reason why we opted for it.

The radiative mesh has been set with respect to the nil values of function F. For the speed up of computations, one can approximate the position of the radiative surface to the closest nodes of the interface (Figure 4).

Solution algorithm. The velocity and temperature problems are solved sequentially at each step by a Newton Raphson method and an implicite Eulerian temporal scheme.

Example of result

In the presented example fluid is air. Rotor and stator are made of steel. The rotation velocity is 0,2Rad/s. The heat flux brought back to the minimum radius of the rotor is 100W/m2. The mesh is composed of 640 linear elements. The cooling fluid maintains the outer side of the stator at 500°C. We present the obtained results for two positions of the rotor.

In Figures 5 and 6 one can see the influence of the presence of two notches. Whenever the notches of the rotor and the stator are crossing, some fluid is moving from one notch to the other. This phenomenon, which is confirmed by the velocity profile (not presented), largely increases the exchange between rotor and stator. The global exchange coefficient between rotor and stator is improved by the interaction of both notches. In Figure 7 we represented the Nusselt number evolution with Taylor Number (24) for two configurations, one with the notched rotor and stator and the other smoothed.

$$Ta = \frac{\omega R_i^{1/2} (e_{eq})^{3/2}}{v} \qquad (24)$$

In the notched configurations, the transfer is improved as the Taylor number is raising and seems to stabilize for Ta = 100. We could not model Ta number over 100 because our model is laminar. In the notched case, the raising velocity

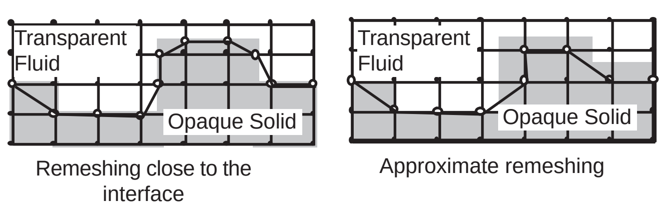

Figure 4. Mesh of the opaque moving radiating surface

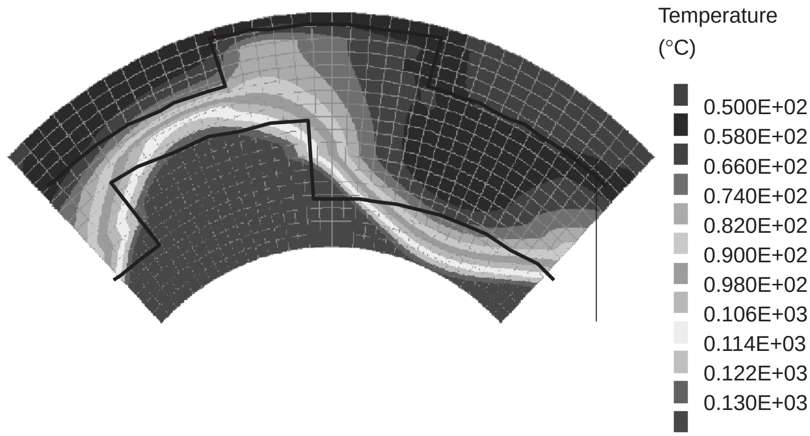

Figure 5.
Temperature field when rotor and stator gaps are facing

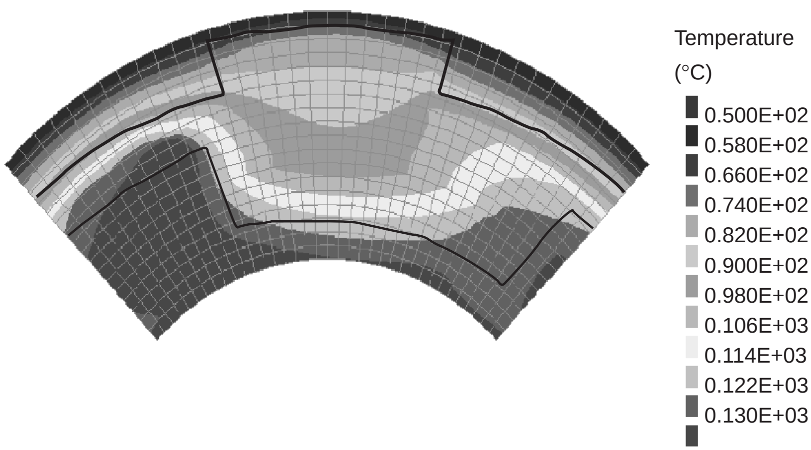

Figure 6.
Temperature field when rotor and stator gaps are opposed

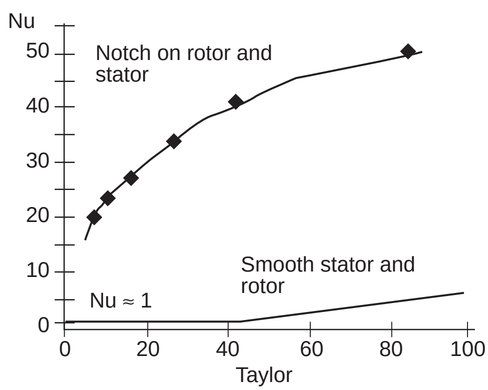

Figure 7.
Comparison of heat transfer between rotor and stator for notched and smooth configurations

of the rotor improves the exchange. On the contrary, for smoothed rotor and stator the transfer is almost purely conductive until Ta = 50 and raises slowly and linearly after this point.

Conclusion

We have developed a finite element model for radiative transfer between grey diffuse surfaces coupled with diffusive transfers. The methods, which have been used for the finite element discretization and for the computation of form factors, aim at adapting the computation effort according to the required accuracy. The approaches lead to a finite element model fast enough to deal with the moving surface problems. 3D geometries are nevertheless limited by the computations or face to face shadows.

The evolution of surfaces and different media on a fixed mesh have a wide range of industrial applications. The heat transfer problem in the air gap of an electrical engine has demonstrated the efficiency of this approach.

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